Infinite Sums

Note Title 1/26/2010

Suppose S is a set and $f: S \longrightarrow \mathbb{R}$ is a function. What do we mean by $\sum_{x \in S} f(x)$?

If $S = \{1, 2, 3, \ldots\} = \mathbb{N}$, we might mean $\sum_{n=1}^{\infty} f(n)$ if the series converges. We might also mean

$$\sum_{k=1}^{\infty} f(2k) + \sum_{k=0}^{\infty} f(2k+1). \quad \left(\sum_{\text{even terms}} + \sum_{\text{odd terms}}\right)$$

But this might not make sense; for example if $f(n) = (-1)^{n+1}/n$.

Here's an appriate theory - the theory of <u>Counting measure</u>. Let $f: S \longrightarrow \mathbb{R}^+$. Let's assume S is countable, Let $A \subset S$, We'll define

$$\sum_{x \in A} f(x) = \int_A f = \sup\left\{ \sum_{x \in F} f(x) : F \text{ finite}, F \subset A \right\}.$$

It might happen that $\int_A f = +\infty$. Notice by assumption $\int_A f \geq 0$, since $f(x) \geq 0$, all x.

In this note I will write $\int_A f$ instead of $\sum_{x \in A} f(x)$, since I don't want A to be ordered. If $f = \chi_A$, the characteristic function of A

defined by $\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$, we'll set

$\mu(A) = \int_S \chi_A$. Then μ is a <u>countably additive measure</u> on the set of all subsets of S. This means $\mu(\cup A_j) = \sum \mu(A_j)$ when $A_j \cap A_k = \phi$ for all pairs j, k, with $j \neq k$. If A is a finite set $\mu(A) = |A|$, the number of elements of A.

If $\int_S f$ is finite we will say that f is <u>integrable</u>, or <u>summable</u>.

Suppose we <u>order</u> the elements of S. That means we have a 1-1, onto map $\sigma : N \longrightarrow S$, where $N = \{1, 2, 3, \cdots, \}$ the set of natural numbers. Let $f: S \to R^+$. We get a sequence $a_n = f(\sigma(n))$.

Now recall the definition of $\int_S f = \sup\left\{ \sum_{x \in F} f(x) : F \subset S, F \text{ finite} \right\}$

Take any finite set F. Then $F \subset \{\sigma(1), \cdots, \sigma(n)\}$ for large enough n. So

$$\sum_{x \in F} f(x) \leq \sum_1^n a_n \leq \int_S f .$$

Hence $\quad \sum_{x\in F} f(x) \le \sum_1^\infty a_n \le \int f.$

This implies that

$$\sup\left\{\sum_{x\in F} f(x) : F\subset S, F \text{ finite}\right\} = \int f = \sum_1^\infty a_n .$$

This is a version of Riemann's rearrangement theorem for absolutely convergent series. Now it is a theorem of measure theory (easy to prove in this case) that if $S = \cup A_j$ is a disjoint union then

(1) $$\int_S f = \sum \int_{A_j} f$$

In particular this says that if we order S and let A be the (infinite) set with even indices and B be the set with odd indices

$$\sum_1^\infty a_n = \int f = \int_A f + \int_B f = \left(\sum_{k=1}^\infty a_{2k}\right) + \left(\sum_{k=0}^\infty a_{2k+1}\right),$$

we can sum separately and then add. This is a new result.

Here is a proof of (1). First we prove that if $\quad A = A_1 \cup A_2, \quad A_1 \cap A_2 = \phi$ (A_j maybe infinite).

(2) $\int_A f = \int_{A_1} f + \int_{A_2} f$. By definition $\int_{A_j} f = \int_S \chi_{A_j} \cdot f$.

It's easy to see that

$$\int_S f \le \int_S g \quad \text{if } 0 \le f \le g.$$

Let F be any finite set. Then

$$\int_A f = \int_S \chi_A f = \sup\left\{ \sum_{x \in F} \chi_A f(x) : F \text{ finite} \right\}$$

$$= \sup\left\{ \sum_{x \in F} f(x) : F \text{ finite}, F \subseteq A \right\}.$$

Any finite set $F \subset A$ can be written uniquely as $F = F_1 \cup F_2$, $F_i = F \cap A_i$. So

$$\sum_{x \in F} f(x) = \sum_{x \in F_1} f(x) + \sum_{x \in F_2} f(x) \le \int_A f$$

where $F \subset A$. So

$$\sum_{x \in F} f(x) \le \sum_{x \in F_1} f(x) + \sup\left\{ \sum_{x \in F_2} f(x) : \begin{array}{l} F_2 \text{ finite} \\ F_2 \subset A_2 \end{array} \right\}$$

$$= \sum_{x \in F_1} f(x) + \int_{A_2} f$$

$$\le \int_{A_2} f$$

Now take sup of first term on right and get

$$\sum_{x\in F} f(x) \le \int_{A_1} f + \int_{A_2} f \le \int_A f$$

For any finite $F \subset A$. Finally take sup on left to get

$$\int_A f \le \int_{A_1} f + \int_{A_2} f \le \int_A f \qquad \text{QED}$$

<u>Prop</u> Let $A = A_1 \cup A_2 \cup \cdots \cup A_n$, $A_i \cap A_j = \phi$, if $i \ne j$.

(3) Then $$\int_A f = \int_{A_1} f + \cdots + \int_{A_n} f$$

proof: use induction and (2).

<u>Theorem</u>: Let $A = \bigcup_1^\infty A_j$, $A_i \cap A_j = \phi$, if $i \ne j$.

Then

(4) $$\int_A f = \sum_1^\infty \int_{A_j} f$$

(the right side is an infinite series)

<u>Proof</u>: Let F be any finite set. Then for some (finite) index $F \subset A_1 \cup \cdots \cup A_N$.

$$\int_{A_1 \cup \cdots \cup A_N} f = \sum_{j=1}^N \int_{A_j} f.$$

$$\sum_{x\in F} f(x) \le \int_{A_1\cup\cdots A_N} f = \sum_{j=1}^{N} \int_{A_j} f \le \int_A f, \text{ by } (3)$$

since $\chi_{A_1\cup\cdots\cup A_N} f \le \chi_A f$. (Here we use

$\int f \le \int g$ if $f \le g$.)

Now take sup over N,

$$\sum_{x\in F} f(x) \le \sum_{j=1}^{\infty} \int_{A_j} f \le \int_A f;$$

and then sup over F:

$$\int_A f \le \sum_{j=1}^{\infty} \int_{A_j} f \le \int_A f \qquad \text{Q.E.D.}$$

<u>Cor</u>: Suppose $a_{n,m} \ge 0$ and $\sum a_{n,m}$ converges. Then $\sum a_{n,m} = \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} a_{n,m}\right)$. Also if $\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} a_{n,m}\right)$ converges, so does $\sum a_{n,m}$ and

$$\sum a_{n,m} = \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} a_{n,m}\right).$$

<u>proof</u>: This follows since, $\sum_j \int_{A_j} f = \int_A f$ means that if one is finite, so is the other.